

The Gauss–Lucas Theorem

Proof of the Theorem

Lecture Notes

September 24, 2026

Theorem (Gauss–Lucas)

Let $z_1, \dots, z_J$ be the (distinct) zeros of a polynomial p . Then all zeros of $p'(z)$ lie inside the convex hull of $\{z_1, \dots, z_J\}$.

Proof of the Theorem. Writing $p(z) = a_d \prod_{j=1}^J (z - z_j)^{m_j}$, logarithmic differentiation gives

$$\frac{p'(z)}{p(z)} = \sum_{j=1}^J \frac{m_j}{z - z_j} = \sum_j m_j \frac{\overline{z - z_j}}{|z - z_j|^2}.$$

If z_0 is itself a multiple root of p , there is nothing to prove: $z_0 \in \{z_1, \dots, z_J\} \subset H$. (By the Proposition established when discussing roots and multiplicity (in the companion note), a common root of p and p' is automatically a multiple root of p – a simple root of p has order 0, i.e. is not a root at all, in p' – so this covers every root z_0 of p' with $p(z_0) = 0$.) Otherwise, suppose z_0 is not a root of p , but is a root of p' . Then $p'(z_0)/p(z_0) = 0$, i.e.,

$$\sum_j m_j \frac{\overline{z_0 - z_j}}{|z_0 - z_j|^2} = 0, \quad \text{so} \quad \bar{z}_0 = \sum_j \mu_j \bar{z}_j, \quad \mu_j := \frac{m_j/|z_0 - z_j|^2}{\sum_i m_i/|z_0 - z_i|^2}.$$

Conjugating, $z_0 = \sum_j \mu_j z_j$ with $\mu_j \geq 0$ and $\sum_j \mu_j = 1$. So $z_0 \in H$. □